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A polyhedral study on 0–1 knapsack problems with disjoint cardinality constraints: Strong valid inequalities by sequence-independent lifting[☆]

Bo Zeng[a,*], Jean-Philippe P. Richard[b,1]

[a] Department of Industrial and Management Systems Engineering, University of South Florida, Tampa, FL 33620, USA

[b] Department of Industrial and Systems Engineering, University of Florida, Gainesville, FL 32611, USA

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ABSTRACT

We study the set of 0–1 integer solutions to a single knapsack constraint and a set of non-overlapping cardinality constraints (MCKP), which generalizes the classical 0–1 knapsack polytope and the 0–1 knapsack polytope with generalized upper bounds. We derive strong valid inequalities for the convex hull of its feasible solutions using sequence-independent lifting. For problems with a single cardinality constraint, we derive two-dimensional superadditive lifting functions and prove that they are maximal and non-dominated under some mild conditions. We then show that these functions can be used to build strong valid inequalities for problems with multiple disjoint cardinality constraints. Finally, we present preliminary computational results aimed at evaluating the strength of the cuts obtained from sequence-independent lifting with respect to those obtained from sequential lifting.

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1. Introduction

In this paper, we study the 0–1 knapsack model (KP) with a set of disjoint (non-overlapping) cardinality constraints (MCKP) that was also investigated in [1] using sequential lifting. Specifically, given a partition $(N_0, N_1, \ldots, N_r)$ of $N = \{1, \ldots, n\}$ and $R = \{1, \ldots, r\}$, we study

$$\hat{S}_r = \left\{ x \in \{0,1\}^n : \sum_{j \in N_0} \hat{a}_j x_j + \sum_{i \in R} \sum_{j \in N_i} \hat{a}_j x_j \le \hat{b},\ \sum_{j \in N_i} x_j \le \hat{K}_i,\ i \in R \right\}, \tag{1}$$

or, the following equivalent set obtained by complementing the variables that have negative knapsack coefficients,

$$S_r = \left\{ x \in \{0,1\}^n : \sum_{j \in N_0} a_j x_j + \sum_{i \in R} \sum_{j \in N_i} a_j x_j \le b,\ \sum_{j \in N_i^+} x_j - \sum_{j \in N_i^-} x_j \le K_i,\ \forall i \in R \right\} \tag{2}$$

where $b \in \mathbb{R}$, $a_j \in \mathbb{R}_+$ for $j \in N$, $K_i \in \mathbb{Z}$, $N_i^+ = \{j \in N_i : \hat{a}_j \ge 0\}$ and $N_i^- = \{j \in N_i : \hat{a}_j < 0\}$ for $i \in R$. We assume that $b \ge 0$ (since otherwise, $S_r = \emptyset$), $a_j \le b$ for $j \in N$ (since otherwise $x_j = 0$) and $|N_i^-| \ge |K_i|$ if $K_i < 0$ (since otherwise $S_r = \emptyset$). We denote the convex hull of S_r by PS_r. Furthermore, when $N_i^- = \emptyset$ for $i \in R$, we denote the set S_r as S_r^+ and its convex hull as PS_r^+. Because S_r is a set of finite cardinality, it can easily be shown that PS_r is a polytope.

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* Corresponding author. Tel.: +1 813 974 5588.
E-mail addresses: bzeng@eng.usf.edu (B. Zeng), richard@ise.ufl.edu (J.-P.P. Richard).

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As we described in [1] and the references therein, MCKP has both theoretical and practical significance. However, little work has been devoted to the study of the polyhedral structure of $P\hat{S}_r$. Exceptions include the work of Glover and Sherali [2,3] and our work in [1]. The research we perform in this paper is different from the work of Glover and Sherali and the work presented in [1] in that we develop sequence-independent lifting tools for PS_r and derive closed-form lifted inequalities.

The paper is organized as follows. In Section 2, we discuss general conditions under which lifting is sequence-independent. In particular, we describe a subset of superadditivity conditions that already implies that lifting is sequence-independent. In Section 3, we briefly review some results about the *generalized cover inequalities* introduced in [1] and propose a transformation that allows the generation of valid inequalities for PS_r from known valid inequalities for PS_r^+. In Section 4, we derive a set of superadditive lifting functions for PS_1^+ and prove that they are non-dominated and maximal. In Section 5, we present $(r+1)$-dimensional non-dominated and maximal superadditive lifting functions for generalized cover inequalities in PS_r^+. In Section 6, we describe the results of a computational experiment that indicate that our cutting planes could be useful in branch-and-cut algorithms for MIPs.

We believe that this paper has the following two major contributions. To the best of our knowledge, it is the first work where a multi-dimensional superadditive lifting function is derived and proven to be strong for a general relaxation of an unstructured 0–1 MIP set with multiple constraints. Second, it provides a unified view and strictly generalizes the known polyhedral results for KP and KP with generalized upper bounds (GUBKP).

2. Lifting in PS_r

In this section, we give a brief review on how MIP lifting can be used to generate strong valid inequalities for PS_r. In Section 2.1, we present basic lifting definitions and results from [1]. In Section 2.2, we describe the general superadditive conditions introduced by Wolsey [4] under which lifting is easier to perform. Because of the specific nature of PS_r, we show that it is sufficient to consider a subset of these conditions in our case. This result permits an easier derivation of stronger lifting coefficients for PS_r in the following sections. Finally, in Section 2.3, we discuss a procedure to generate strong inequalities when superadditive conditions are not naturally satisfied.

2.1. Basic lifting results

In this section, we briefly review the necessary concepts and basic results related to lifting. Consider the 0–1 integer set

$$G = \left\{ x \in \{0,1\}^n : \sum_{j\in N} A_j x_j \le b \right\} \tag{3}$$

where $A_j \in \mathbb{R}^m$ for $j \in N$ and $b \in \mathbb{R}^m$. Denote the convex hull of G as PG and consider the restriction $PG(N')$ of PG obtained by fixing a subset N' of variables to 0, i.e., define $PG(N') := \text{conv}\{G \cap \{x \in \mathbb{R}^n : x_j = 0\ \forall j \in N'\}\}$ where $N' = \{1, \dots, n'\} \subseteq N$. Sequential lifting is the process by which the *seed inequality*,

$$\sum_{j\in N\setminus N'} \alpha_j x_j \le \alpha_0, \tag{4}$$

which is assumed to be valid for $PG(N')$, is converted into a valid inequality for PG of the form

$$\sum_{j\in N} \alpha_j x_j \le \alpha_0 \tag{5}$$

by reintroducing the variables $x_1, \dots, x_{n'}$ in (4) one at a time. For $i \in N'$, the lifting coefficients α_i in (5) can be derived from the lifting functions

$$\begin{aligned} f_i(\vec{Z}) = \min\ & \alpha_0 - \sum_{j\in N\setminus N'} \alpha_j x_j - \sum_{1\le j<i} \alpha_j x_j \\ \text{s.t.}\ & \sum_{j\in N\setminus N'} A_j x_j + \sum_{1\le j<i} A_j x_j \le b - \vec{Z} \\ & x_j \in \{0,1\}\ \forall j \in N \setminus N' \cup \{1,\dots,i-1\} \end{aligned} \tag{6}$$

where $\vec{Z} \in \mathbb{R}^m$ and $f_i(\vec{Z}) = +\infty$ if (6) is infeasible.

In the remainder of this paper, we will use f to denote f_1 and will refer to f as the (*exact*) *lifting function* of (4). Next, we state a basic result from [1] that has direct application in this paper.

Proposition 1 ([1]). *Let $i, j \in N'$ be such that $i \le j$ and let $\vec{Y}, \vec{Z} \in \mathbb{R}^m$ be such that $\vec{Y} \le \vec{Z}$. Assume also that $f_i(\vec{Y}) < +\infty$ and $f_i(\vec{Z}) < +\infty$. Then*

(i) $f_i(\vec{0}) \ge 0$,
(ii) $f_i(\vec{Y}) \le f_i(\vec{Z})$,
(iii) $f_i(\vec{Y}) \ge f_j(\vec{Y})$. $\square$

2.2. Sequence-independent lifting

Lifted inequalities generated by setting the lifting coefficients α_i to $f_i(A_i)$ (whenever $f_i(A_i) < +\infty\ \forall i$) are strong. However, the amount of computation needed to obtain them is often prohibitive as a different optimization problem must be solved for every variable that is lifted. Fortunately this computational burden can be significantly reduced if the exact lifting function f is well-structured.

Definition 1. Let $\mathbb{D} \subseteq \mathbb{R}^n$. A function $g : \mathbb{D} \mapsto \mathbb{R}$ is superadditive if $g(x) + g(y) \le g(x + y)$ for all $x, y, x + y \in \mathbb{D}$. $\square$

Wolsey [4] shows that, for the 0–1 knapsack problems, all lifting coefficients can be directly obtained from f when f is superadditive over an appropriate domain. Gu et al. [5] generalize Wolsey's results to 0–1 mixed integer programs. Atamturk [6] extends the results to general mixed integer programs. Because of its computational advantages, the superadditive lifting theory has been used in various applications to derive strong inequalities for MIPs. As an example, Marchand and Wolsey [7] use superadditive lifting to derive two families of closed-form facet-defining inequalities for 0–1 knapsack sets with a single continuous variable.

Although the condition that f is superadditive over $\mathbb{D} = \mathbb{R}^m$ is sufficient for sequence-independent lifting to hold, there are weaker conditions that still imply sequence-independent lifting. We give such conditions next.

Theorem 2. *Assume that $f(A_i) + f\left(\sum_{j\in T} A_j\right) \le f\left(\sum_{j\in T\cup\{i\}} A_j\right)$ for all $i \in N'$ and for all $T \subseteq N' \setminus \{i\}$. Assume also that $f(A_i) < +\infty$ for all $i \in N'$. Then $f_i(A_t) = f(A_t)$ for $i \in N'$ and for $t = i, \dots, n'$.*

Proof. The proof is by induction. The result is obvious for $i = 1$. Assume now that we have already established that for $i \le k$ where $1 \le k \le n' - 1$, $f_i(A_j) = f(A_j)$ for $j = i, \dots, n'$. We will prove that the result holds for $i = k + 1$.

We know from Proposition 1 that $f_{k+1}(A_t) \le f(A_t)$ for $t = k + 1, \dots, n'$. Further, since $x_j \in \{0, 1\}$ for $1 \le j \le k$, we observe from (6) that $f_{k+1}(A_t) = \min_{\{T\subseteq\{1,\dots,k\}:\sum_{j\in T\cup\{t\}} A_j\le b\}} \left\{f\left(\sum_{j\in T} A_j + A_t\right) - \sum_{j\in T} f_j(A_j)\right\}$. From the inductive hypothesis, we have $\sum_{j\in T} f_j(A_j) = \sum_{j\in T} f(A_j)$ since $T \subseteq \{1, \dots, k\}$. From the theorem assumption on f, it can easily be verified that $\sum_{j\in T} f(A_j) \le f\left(\sum_{j\in T} A_j\right)$ for $T \subseteq \{1, \dots, k\}$. So, $f_{k+1}(A_t) \ge \min_{T\subseteq\{1,\dots,k\}} \left\{f\left(\sum_{j\in T} A_j + A_t\right) - f\left(\sum_{j\in T} A_j\right)\right\}$ for $t = k+1, \dots, n'$. Again, from the theorem assumption on f, we conclude that $f_{k+1}(A_t) \ge f(A_t)$. This shows that $f_{k+1}(A_t) = f(A_t)$ for $t = k + 1, \dots, n'$. $\square$

Theorem 2 shows that it is sufficient to prove that f is superadditive over a suitable subset of $\mathbb{R}^m$ to ensure sequence-independent lifting. In many cases, the size of this subset increases exponentially with the number of variables to be lifted. As a result, it becomes more convenient to verify that f is superadditive over $\mathbb{R}^m$ or $\mathbb{R}^m_+$. However, the number of conditions to verify for a problem with cardinality constraints remains manageable. This is because all the variable coefficients in the cardinality constraints are either -1, 0 or 1. By combining this observation with the result of Theorem 2, we derive next the simpler conditions for sequence-independent lifting in PS_r. To simplify the presentation, we define for $N^0, N^1 \subseteq N$ with $N^0 \cap N^1 = \emptyset$,

$$PS_r(N^0, N^1) := \text{conv}\{x \in S_r : x_j = 0,\ \forall j \in N^0,\ \text{and } x_j = 1,\ \forall j \in N^1\}.$$

We define $PS_r^+(N^0, N^1)$ similarly.

Proposition 3. *Let f be the lifting function of a valid inequality of $PS_r(N', \emptyset)$. Then, the lifting of variables in N' is sequence-independent if*

$$f\begin{pmatrix} y \\ \vec{I} \end{pmatrix} < +\infty \quad \text{and} \quad f\begin{pmatrix} y \\ \vec{I} \end{pmatrix} + f\begin{pmatrix} z \\ \vec{h} \end{pmatrix} \le f\begin{pmatrix} y+z \\ \vec{I}+\vec{h} \end{pmatrix} \tag{7}$$

(i) $\forall (y, \vec{I}) \in [0, b] \times \{\vec{0}, e_1, \dots, e_r\}$ *and* $\forall (z, \vec{h}) \in [0, b] \times \mathbb{Z}^r_+$ *for* PS_r^+,

(ii) $\forall (y, \vec{I}) \in [0, b] \times \{\vec{0}, \pm e_1, \dots, \pm e_r\}$ *and* $\forall (z, \vec{h}) \in [0, b] \times \mathbb{Z}^r$ *for* PS_r,

where $e_1, \dots, e_r$ are the unit vectors of $\mathbb{R}^r$. $\square$

Verifying that a function is superadditive is often cumbersome, even for one-dimensional functions. Proposition 3 will be used extensively in the following sections as it reduces the number of conditions to verify to guarantee that lifting is sequence-independent in PS_r. Note that a lifting function satisfying the conditions of Proposition 3 is not necessarily superadditive over $[0, b] \times \mathbb{Z}^r$. However, in the remainder of this paper, we will refer to functions satisfying (7) as superadditive.

2.3. Approximate superadditive lifting

It is possible that the exact lifting function of a seed inequality of interest will not satisfy the conditions of Theorem 2. In such a situation, lower approximations of the exact lifting functions that satisfy the conditions of Theorem 2 can be used to obtain strong valid inequalities without having to solve lifting problems repeatedly. This idea was first used by Gu et al. [8,5] to derive efficient lifting procedures for 0–1 knapsack and single node flow problems. Applications of the idea are also given by Atamturk [9] for general mixed-integer knapsack sets and by Shebalov and Klabjan [10] for mixed-integer programs with variable upper bounds.

For any given lifting function, Gu et al. [5] give a constructive proof of the existence of a superadditive lower approximation. In practice, there are usually many such approximations. Therefore, evaluating the quality of a proposed approximation is important. To measure the strength of a superadditive approximation, Gu et al. [8,5] propose two criteria: non-dominance and maximality. Here, we summarize these concepts and present them for higher dimensions. Thereafter, we refer to the exact lifting function as f and refer to its superadditive approximation as g. We denote the domain of f by $\mathbb{X} \subseteq \mathbb{R}^m$.

Let $\underline{\mathbb{X}} \subseteq \mathbb{X}$ be a set containing the coefficients of all the variables to be lifted; i.e. $A_i \in \underline{\mathbb{X}}$ for $i \in N'$. Because the superadditive approximation g of f will only be used over the subset $\underline{\mathbb{X}}$ of its domain, it is not important that the approximation is strong over $\mathbb{X} \setminus \underline{\mathbb{X}}$. In PS_r, for example, we only care about the strength of the superadditive approximation over $\underline{\mathbb{X}} = [0, b] \times \{\vec{0}, \pm e_1, \ldots, \pm e_r\}$ since all the lifting coefficients will be obtained by evaluating the lifting function in this range. This observation motivates the following definitions.

Definition 2. For a given exact lifting function $f(x)$ defined over $\mathbb{X}$, we say that $g(x)$ is a non-dominated superadditive approximation of $f(x)$ if

(i) $g(x) \le f(x)\ \forall x \in \mathbb{X}$;
(ii) $g(x)$ is a superadditive over $\mathbb{X}$, and
(iii) there is no other superadditive lower approximation $g'(x)$ of $f(x)$ such that $g(x) \le g'(x)$ for $x \in \underline{\mathbb{X}}$ and $g(x_0) < g'(x_0)$ for some $x_0 \in \underline{\mathbb{X}}$. □

Another desirable property for approximate lifting functions is that they yield the same lifting coefficients as sequential lifting for those variables whose lifting coefficients do not depend on the lifting sequence. This requirement motivates the following definition of maximal set and maximal superadditive approximation.

Definition 3. Let $\underline{\mathbb{X}}$ be a subset of $\mathbb{X}$. We say that

$$E = \{x \in \underline{\mathbb{X}} : f_i(x) = f_1(x) \text{ for all choices of lifting orders and for all choices of } A_i \in \underline{\mathbb{X}}, \text{ where } i \in N'\}$$

is the maximal set of the lifting function f_1 with respect to $\underline{\mathbb{X}}$. When the set $\underline{\mathbb{X}}$ is clear from the context, we simply say that E is the maximal set of f_1. □

Definition 4. Given a valid superadditive approximation $g(x)$ of $f(x)$, we say that $g(x)$ is maximal over $\underline{\mathbb{X}}$ if $g(x) = f(x)$ for all $x \in E$. □

Clearly, non-dominated and maximal superadditive approximations are most suitable for approximate lifting. We introduce next a definition to describe inequalities that are obtained using non-dominated and maximal superadditive approximations.

Definition 5. Let f be the lifting function of the seed inequality (4). We say that (5) is a maximal inequality if $\alpha_j = g(A_j)$ for $j \in N'$ where g is a non-dominated and maximal superadditive approximation of f. □

3. Deriving cuts for PS_r from cuts for PS_r^+

To obtain strong inequalities using sequence-independent lifting, it is necessary to first derive strong seed inequalities. In [1], we introduce a family of facet-defining *generalized cover inequalities* (GCIs) that can be used as seed inequalities in lifting procedures. Unfortunately, GCIs in PS_r are generally not well-structured and deriving strong superadditive approximations of their lifting functions is difficult. In the case of PS_r^+, however, we prove in [1] that GCIs reduce to classical minimal cover inequalities. We present this result in Corollary 4.

Corollary 4 ([1]). *Let C be a minimal cover of the knapsack constraint, let $C_i = C \cap N_i$ and let $\eta_i = K_i - |C_i|$. Then, the cover inequality*

$$\sum_{j \in C} x_j \le |C| - 1 \tag{8}$$

is a generalized cover inequality for PS_r^+ that is facet-defining for $PS_r^+(N \setminus C, \emptyset)$ if and only if one of the following conditions is satisfied

(i) *$C_i = \emptyset$ for all $i \in R$, i.e. $C \subseteq N_0$;*
(ii) *$C = C_i$ for some $i \in R$ and $\eta_i \ge -1$;*
(iii) *$C \ne C_i, \cup_{i=1}^r C_i \ne \emptyset$ and $\eta_i \ge 0$ for all $i \in R$.* □

Observe that in the case of GUBKP, condition (iii) corresponds to the notion of minimal GUB cover used by Vance and Nemhauser [11], Sherali and Lee [12], and Wolsey [13]. We further note in [1] that the exact lifting function of (8) in PS_r^+ has a structure similar to that of the lifting function of the minimal cover inequality in KP. Therefore, we focus only on developing superadditive lifting functions and studying sequence-independent lifting in PS_r^+ in the remainder of this paper. This assumption is not completely restrictive as there is a nontrivial transformation that allows us to convert valid inequalities for PS_r^+ into valid inequalities for PS_r. We present this transformation next.

For convenience in the exposition, we first rewrite the set $\hat{S}_r$ in (1) as

$$\hat{S}_{r0} = \left\{ x \in \{0,1\}^n : \sum_{i \in R \cup \{0\}} \sum_{j \in N_i} \hat{a}_j x_j \le \hat{b}, \sum_{j \in N_i} x_j \le \hat{K}_i, \ \forall i \in R \right\}. \tag{9}$$

Because the variables in N_0 are not restricted by cardinality constraints, we assume that $\hat{a}_j \ge 0$ for $j \in N_0$. We define $R^- = \{i \in R : \exists j \in N_i, \hat{a}_j < 0\}$ and let $R^+ = R \setminus R^-$ and $R_0^+ = R^+ \cup \{0\}$. We also assume that $R^- \neq \emptyset$ since otherwise $\hat{S}_{r0}$ reduces to S_r^+. In each cardinality constraint of R^-, we introduce $\hat{K}_i$ binary variables $y_{i,j}$'s that will act as the slack variable. We obtain:

$$\hat{S}_{r1} = \left\{ (x,y) \in \{0,1\}^{n+\sum_{i \in R^-} \hat{K}_i} : \sum_{i \in R \cup \{0\}} \sum_{j \in N_i} \hat{a}_j x_j \le \hat{b}, \sum_{j \in N_i} x_j \le \hat{K}_i, \ \forall i \in R^+, \sum_{j \in N_i} x_j + \sum_{j=1}^{\hat{K}_i} y_{i,j} = \hat{K}_i, \ \forall i \in R^- \right\}.$$

Clearly, $\mathrm{proj}_x(\hat{S}_{r1}) = \hat{S}_{r0}$. For each $i \in R^-$, we choose $j_i^* \in \arg\max\{-\hat{a}_j : j \in N_i^-\}$ and define $a_i^* = -\hat{a}_{j_i^*}$. Next, we multiply each cardinality constraint i in R^- by $\hat{a}_i^*$ and add the resulting equalities to the knapsack constraint. We obtain:

$$\hat{S}_{r2} = \left\{ (x,y) \in \{0,1\}^{n+\sum_{i \in R^-} \hat{K}_i} : \sum_{i \in R_0^+} \sum_{j \in N_i} \hat{a}_j x_j + \sum_{i \in R^-} \left(\sum_{j \in N_i} (\hat{a}_j + \hat{a}_i^*) x_j + \hat{a}_i^* \sum_{j=1}^{\hat{K}_i} y_{i,j} \right) \le b', \right.$$
$$\left. \sum_{j \in N_i} x_j \le \hat{K}_i, \ \forall i \in R^+, \sum_{j \in N_i} x_j + \sum_{j=1}^{\hat{K}_i} y_{i,j} = \hat{K}_i, \ \forall i \in R^- \right\}$$

where $b' = \hat{b} + \sum_{i \in R^-} \hat{a}_i^* \hat{K}_i$. Because of the definition of $\hat{a}_i^*$, it is clear that $\ddot{a}_j = \hat{a}_j + \hat{a}_i^* \ge 0$ for all $j \in N_i$ and $i \in R^-$. Finally, after removing $x_{j_i^*}$ from its corresponding cardinality constraint, we obtain:

$$\hat{S}_{r3} = \left\{ (x,y) \in \{0,1\}^{n+\sum_{i \in R^-} (\hat{K}_i - 1)} : \sum_{i \in R_0^+} \sum_{j \in N_i} \hat{a}_j x_j + \sum_{i \in R^-} \left(\sum_{j \in N_i \setminus \{j_i^*\}} \ddot{a}_j x_j + \hat{a}_i^* \sum_{j=1}^{\hat{K}_i} y_{i,j} \right) \le b', \right. \tag{10a}$$

$$\sum_{j \in N_i} x_j \le \hat{K}_i, \ \forall i \in R^+ \tag{10b}$$

$$\sum_{j \in N_i \setminus \{j_i^*\}} x_j + \sum_{j=1}^{\hat{K}_i} y_{i,j} \le \hat{K}_i, \ \forall i \in R^- \tag{10c}$$

$$\left. \sum_{j \in N_i \setminus \{j_i^*\}} x_j + \sum_{j=1}^{\hat{K}_i} y_{i,j} \ge \hat{K}_i - 1, \ \forall i \in R^- \right\}. \tag{10d}$$

Observe finally that the set $\hat{S}_{r4}$ obtained by considering the constraints (10a)–(10c) is of the form of S_r^+.

We next show how valid cutting planes for $\hat{S}_{r4}$ (such as the higher-order cover inequalities of [2,3] and the lifted generalized cover inequalities described in the following sections) can be used as valid inequalities for $\hat{S}_{r0}$. Observe that this conversion is not direct since $\hat{S}_{r4}$ is not defined in the same space as $\hat{S}_{r0}$. Consider such a valid inequality for $\hat{S}_{r4}$ and denote it as

$$\sum_{i \in R^+} \sum_{j \in N_i} \alpha_j x_j + \sum_{i \in R^-} \left(\sum_{j \in N_i \setminus \{j_i^*\}} \alpha_j x_j + \sum_{j=1}^{\hat{K}_i} \beta_{i,j} y_{i,j} \right) \le \delta. \tag{11}$$

For any fixed i, the variables $y_{i,j}$ play symmetrical roles because they have identical coefficients in the knapsack and in the cardinality constraints. Therefore, any inequality obtained from (11) by a permutation of the coefficients of the variables $y_{i,j}$

is valid for $\hat{S}_{r4}$. After summing these inequalities for all $i \in R^-$ and $j \in N_i$, and scaling the resulting inequality, we obtain that

$$\sum_{i\in R^+}\sum_{j\in N_i}\alpha_j x_j + \sum_{i\in R^-}\left(\sum_{j\in N_i\setminus\{j_i^*\}}\alpha_j x_j + \beta_{i,0}\sum_{j=1}^{\hat{K}_i} y_{i,j}\right) \le \delta \tag{12}$$

where $\beta_{i,0} = \frac{1}{\hat{K}_i}\sum_{j=1}^{\hat{K}_i}\beta_{i,j}$ is valid for $\hat{S}_{r4}$. Observe that in $\hat{S}_{r2}$, we defined $\sum_{j=1}^{\hat{K}_i} y_{i,j} = \hat{K}_i - \sum_{j\in N_i} x_j$. Substituting for $\sum_{j=1}^{\hat{K}_i} y_{i,j}$ in (12), we obtain

$$\sum_{i\in R^+}\sum_{j\in N_i}\alpha_j x_j + \sum_{i\in R^-}\left(\sum_{j\in N_i\setminus\{j_i^*\}}(\alpha_j - \beta_{i,0})x_j - \beta_{i,0}x_{j_i^*}\right) \le \delta - \sum_{i\in R^-}\beta_{i,0}\hat{K}_i \tag{13}$$

which is valid for $\hat{S}_{r0}$. Next, we illustrate in Example 1 that this transformation can generate strong cutting planes for PS_r from strong cutting planes for PS_r^+.

Example 1. Consider

$$S_1 = \{x \in \{0,1\}^6 : 7x_1 - 7x_2 - 6x_3 + 5x_5 + x_6 \le 6, x_2 + x_3 + x_4 + x_6 \le 2\}.$$

Clearly, $j_1^* = 2$ and $\hat{a}_1^* = 7$. By introducing binary slack variables y_1 and y_2 and using the above transformation, we obtain the following set that corresponds to S_{r4}:

$$S_1^+ = \{(x,y) \in \{0,1\}^7 : 7x_1 + x_3 + 7x_4 + 5x_5 + 8x_6 + 7y_1 + 7y_2 \le 20, x_3 + x_4 + x_6 + y_1 + y_2 \le 2\}.$$

We consider the generalized cover inequality

$$x_1 + x_4 + x_6 \le 2. \tag{14}$$

After lifting variables x_3, x_5, y_1 and y_2, we obtain

$$x_1 + x_4 + x_6 + y_1 + y_2 \le 2. \tag{15}$$

Because the coefficients of y_1 and y_2 are identical in (15), it is sufficient to substitute $y_1 + y_2 = 2 - x_2 - x_3 - x_4 - x_6$ in (15) to obtain

$$x_1 - x_2 - x_3 \le 0, \tag{16}$$

which is valid for S_1. It can be verified that (16) is facet-defining for $\text{conv}(S_1)$. □

Although in Example 1 we show a situation where a facet-defining inequality of PS_1^+ is transformed into a facet-defining inequality of PS_1, it is not always the case. However, when all cardinality constraints are GUB constraints, the transformation described above always transforms facets of PS_r^+ into facets of PS_r, as proven by Johnson amd Padberg [14].

Corollary 5 ([14]). *Assume that $\hat{K}_i = 1$ for all $i \in R$. Inequality (12) is facet-defining for* $\text{conv}(\hat{S}_{r4})$ *if and only if (13) is facet-defining for* $\text{conv}(\hat{S}_{r0})$. □

Note that Corollary 5 holds under the weaker assumption that $\hat{K}_i = 1$ for $i \in R^-$. The procedure described above shows that it is possible to obtain valid inequalities for PS_r from strong valid inequalities for PS_r^+. Therefore, in the following sections, we focus only on the study of efficient lifting methods for PS_r^+. In Section 4, we study PS_1^+, the 0–1 knapsack set with a single cardinality constraint, and derive strong 2-dimensional superadditive lifting functions for generalized cover inequalities. In Section 5, we extend our results to PS_r^+ and derive strong $(r+1)$-dimensional superadditive lifting functions. To the best of our knowledge, this is the first time that superadditive lower approximations of multi-dimensional lifting functions are proposed and proven to be strong. In fact, research on approximate superadditive lifting has been limited to one-dimensional lifting functions even when the sets studied have multiple constraints; see [8,5,10].

4. Sequence-independent lifting in PS_1^+

In this section, we show how to lift generalized cover inequalities into strong valid inequalities for PS_1^+ using sequence-independent lifting. We denote the exact lifting function of the generalized cover inequality in PS_1^+ by Θ and denote its superadditive approximation by θ throughout the remainder of this paper. We also denote the coefficient of x_j by $(a_j, l_j) \in [0, b] \times \{0, 1\}$. Since PS_1^+ has a single cardinality constraint with only nonnegative coefficients, we use the notation N^+, C^+ and K to represent $N_1^+ = N_1$, C_1 and K_1 respectively. We assume without loss of generality that $C = \{1, \ldots, |C|\}$ and that the variables x_j for $j \in C$ are sorted in non-increasing order of their knapsack coefficients, i.e. $a_1 \ge a_2 \ge \cdots \ge a_{|C|}$. Finally, we recall from Corollary 4 that for PS_1^+, GCIs are based on minimal covers for the knapsack constraint.

Next, we present the main result that is proven in this section.

Theorem 6. *Let C be a generalized cover and let $\theta_0(z)$ be a valid superadditive approximation of $\Theta\binom{z}{0}$. Then the inequality*

$$\sum_{j\in C} x_j + \sum_{j\in N\setminus C} \theta\binom{a_j}{I_j} x_j \le |C| - 1 \tag{17}$$

where

$$\theta\binom{z}{I} = \begin{cases} \theta_0(z) & \text{if } I = 0 \\ \max\left\{\theta_0(z-a^*)+1, \theta_0(z)\right\} & \text{if } I = 1 \end{cases} \tag{18}$$

and

$$a^* = \begin{cases} a_1 & \text{if } C \ne C^+ \text{ and } |C^+| \le K-1 \text{ or if } C = C^+ \text{ and } |C^+| \le K, \\ \max\{a_j : j \in C^+\} & \text{if } C \ne C^+ \text{ and } |C^+| = K, \\ a_2 & \text{if } C = C^+ \text{ and } |C^+| = K+1 \end{cases} \tag{19}$$

is valid for PS_1^+. In particular, if $\theta_0(z)$ is the non-dominated and maximal superadditive approximation of $\Theta\binom{z}{0}$ proposed by Gu et al. [5] over $[0, b]$ and $\theta_0(z) = -\infty$ when $z < 0$, then the lifted generalized cover inequality (17) is maximal over $[0, b] \times \{0, 1\}$. □

Observe that in Theorem 6 we only give the superadditive approximation of Θ over $[0, b] \times \{0, 1\}$ because all coefficients of variables x_j of PS_1^+ belong to this set. A more complete description of the superadditive approximation is needed to prove sequence independence and is presented in Theorem 12. Next, we present five significant characteristics of the superadditive approximation we propose in Theorem 6. We will elaborate on these characteristics in later sections.

(i) The function $\theta\binom{z}{I}$ is within one unit of the exact lifting function $\Theta\binom{z}{I}$, i.e. $|\Theta\binom{z}{I} - \theta\binom{z}{I}| \le 1$ for all $(z, I) \in [0, b] \times \{0, 1\}$. Therefore, the coefficients of the lifted inequality are close to those obtained using sequential lifting.
(ii) θ is stronger than the traditional single-dimensional superadditive approximation presented by Gu et al. [5] and therefore leads to inequalities stronger than those currently obtained through sequence-independent lifting.
(iii) Because $\theta\binom{z}{I}$ is defined as a simple function of $\theta_0(z)$, the value of $\theta\binom{z}{I}$ can be obtained using roughly the same amount of memory and time as that of $\theta_0(z)$. Therefore, Theorem 6 yields an efficient procedure to generate cuts.
(iv) Because $\theta\binom{z}{I}$ is non-dominated and maximal, Theorem 6 suggests that strong 2-dimensional superadditive approximations of lifting functions can be constructed using strong single-dimensional functions. Further, this construction provides a concise proof of superadditivity and can be recursively applied to build strong multi-dimensional superadditive approximations in PS_r^+.
(v) The construction of $\theta\binom{z}{I}$ from $\theta_0(z)$ can be generalized to other single-dimensional superadditive lifting functions. In [15], we propose a framework that can be used to build high-dimensional superadditive approximations based on low-dimensional superadditive functions and give applications for two different MIP models.

In the remainder of this section, we give a proof of Theorem 6. In Section 4.1, we derive an analytical expression for the exact lifting function of the generalized cover inequality and establish a connection between the multi-dimensional exact lifting function and the single-dimensional exact lifting function. In Section 4.2 we present conditions under which lifting is sequence-independent and use these conditions to derive superadditive approximations that are provably strong.

4.1. Exact lifting function of the generalized cover inequality

First, we introduce the notation $A_i = \sum_{1\le j\le i} a_j$ for $i = 1, \ldots, |C|$, $A_0 = 0$ and $\lambda = A_{|C|} - b$. Because the generalized cover C is also a minimal cover for the knapsack constraint, we conclude that $\lambda > 0$. The ith lifting problem of the generalized cover inequality is

$$\begin{aligned} \Theta_i\binom{z}{h} = \min\ & |C| - 1 - \sum_{j\in C} x_j - \sum_{1\le j\le i-1} \alpha_j x_j \\ \text{s.t. } & \sum_{j\in C} a_j x_j + \sum_{1\le j\le i-1} a_j x_j \le b - z \\ & \sum_{j\in C^+} x_j + \sum_{1\le j\le i-1} I_j x_j \le K - h \\ & x_j \in \{0, 1\} \quad \text{for } j \in C \cup \{1, \ldots, i-1\} \end{aligned} \tag{20}$$

where $\alpha_j = \Theta_j\binom{a_j}{I_j}$ and $(z, h) \in [0, b] \times \mathbb{Z}_+$.

Next, we derive an explicit form for $\Theta\binom{z}{h}$. We proceed in two steps. We first give a closed-form expression for $\Theta\binom{z}{0}$ and then compute $\Theta\binom{z}{h}$ from $\Theta\binom{z}{0}$. Because the lifting function $\Theta\binom{z}{0}$ will be used several times in this paper, we denote it as $\Theta^*\binom{z}{0}$. We note that $\Theta^*\binom{z}{0}$ corresponds to the lifting function of the classical minimal cover inequality. This function was studied by Gu and Nemhauser [5].

Theorem 7 (*Adapted from [5]*)**.** *The exact lifting function $\Theta^*\binom{z}{0}$ of the generalized cover inequality satisfies*

$$\Theta^*\binom{z}{0} = \begin{cases} 0 & \text{if } 0 \le z \le A_1 - \lambda \\ i & \text{if } A_i - \lambda < z \le A_{i+1} - \lambda \quad i = 1, \ldots, |C| - 1 \end{cases} \tag{21}$$

for $z \in [0, b]$. □

Next, we describe in Theorem 8 the exact lifting function Θ. We consider four cases: (i) $C \neq C^+$ and $|C^+| \le K - 1$; (ii) $C = C^+$ and $|C^+| \le K$; (iii) $C \neq C^+$ and $|C^+| = K$; and (iv) $C = C^+$ and $|C^+| = K + 1$. We define $A_i^h = \sum_{j=h-i+1}^{h} a_i$ for $h = 1, \ldots, K$ and $i = 1, \ldots, h$. When $i = 0$, we define $A_0^h = 0$ for all h. When $C \neq C^+$, we denote $C^+ = \{l_1, \ldots, l_{|C^+|}\}$ and assume without loss of generality that $a_{l_1} \ge \cdots \ge a_{l_{|C^+|}}$. Similarly, we define $\hat{A}_i = \sum_{j=1}^{i} a_{l_j}$ and $\hat{A}_i^h = \sum_{j=h-i+1}^{h} a_{l_j}$ for $h = 1, \ldots, K$ and for $i = 0, \ldots, h$.

Theorem 8. *The exact lifting function $\Theta\binom{z}{h}$ of the generalized cover inequality satisfies*

Case i: $C \neq C^+$ *and* $|C^+| \le K - 1$

$$\Theta\binom{z}{1} = \Theta^*\binom{z}{0} \tag{22}$$

for $z \in [0, b]$.

Case ii: $C = C^+$ *and* $|C^+| \le K$

$$\Theta\binom{z}{1} = \Theta^*\binom{z}{0} \tag{23}$$

for $z \in [0, b]$.

Case iii: $C \neq C^+$ *and* $|C^+| = K$

$$\Theta\binom{z}{h} = \begin{cases} h - 1 & \text{if } 0 \le z \le \hat{A}_h - \lambda \\ h & \text{if } \hat{A}_h - \lambda < z < \hat{A}_h \\ \max\limits_{i=0,\ldots,h} \left\{ \Theta^*\begin{pmatrix} z - \hat{A}_i^h \\ 0 \end{pmatrix} + i \right\} & \text{if } \hat{A}_h \le z \le b \end{cases} \tag{24}$$

for $z \in [0, b]$ and $h = 1, \ldots, K$.

Case iv: $C = C^+$ *and* $|C^+| = K + 1$

$$\Theta\binom{z}{h} = \begin{cases} h & \text{if } 0 \le z \le A_{h+1} - \lambda \\ h + 1 & \text{if } A_{h+1} - \lambda < z < A_{h+1} \\ \max\limits_{i=0,\ldots,h+1} \left\{ \Theta^*\begin{pmatrix} z - A_i^{h+1} \\ 0 \end{pmatrix} + i \right\} & \text{if } A_{h+1} \le z \le b \end{cases} \tag{25}$$

for $z \in [0, b]$ and $h = 1, \ldots, K$.

Proof. The results for Cases i and ii follow directly from the definition of generalized cover and from the lifting problem (20). The proof for Case iv is very similar to that of Case iii. Therefore, we only give a proof for Case iii.

Since it is easy to verify the value of $\Theta\binom{z}{h}$ for $z \in [0, \hat{A}_h)$ and $h = 1, \ldots, K$, we only derive the value of $\Theta\binom{z}{h}$ for $z \in [\hat{A}_h, b]$ and $h = 1, \ldots, K$. Fix $z \in [\hat{A}_h, b]$ and $h \in \{1, \ldots, K\}$. Define $T := \{j \in C : j \neq l_i,\ \forall i = 1, \ldots, h\}$ and assume that $T = \{k_1, \ldots, k_{|C|-h}\}$ with $k_1 < \cdots < k_{|C|-h}$. Let s be the only index such that $\hat{A}_h - \lambda + \sum_{j=1}^{s-1} a_{k_j} < z \le \hat{A}_h - \lambda + \sum_{j=1}^{s} a_{k_j}$. It is easy to verify that the solution x^* defined as

$$x_j^* = \begin{cases} 0 & \text{if } j \in \{l_1, \ldots, l_h\} \cup \{k_1, \ldots, k_s\} \\ 1 & \text{if } j \in \{k_{s+1}, \ldots, k_{|C|-h}\} \end{cases}$$

is optimal for $\Theta\binom{z}{h}$ and that $\Theta\binom{z}{h} = h + s - 1$.

First we prove that $\max_{i=0,\ldots,h} \left\{ \Theta^*\binom{z-\hat{A}_i^h}{0} + i \right\} \le \Theta\binom{z}{h}$. For $i \in \{0, \ldots, h\}$, we define $\tilde{x}^i$ as the solution obtained by setting the i variables with largest indices in $\{l_1, \ldots, l_h\}$ to 1, i.e.

$$\tilde{x}_j^i = \begin{cases} x_j^* & \text{if } j \in C \setminus \{l_{h-i+1}, \ldots, l_h\} \\ 1 & \text{if } j \in \{l_{h-i+1}, \ldots, l_h\}. \end{cases}$$

The solution $\tilde{x}^i$ satisfies

$$\sum_{j \in C} a_j \tilde{x}_j^i = \sum_{j \in C \setminus \{l_{h-i+1}, \ldots, l_h\}} a_j x_j^* + \sum_{t=h-i+1}^{h} a_{l_t} \le b - z + \sum_{t=h-i+1}^{h} a_{l_t} = b - z + \hat{A}_i^h$$

and also

$$\sum_{j\in C^+}\tilde{x}_j^i=\sum_{j\in C^+}x_j^*+\sum_{t=h-i+1}^{h}1\le K-h+i\le K.$$

It follows that $\tilde{x}^i$ is a feasible solution for the problem $\Theta^*\binom{z-\hat{A}_i^h}{0}$ and has an objective value of $\Theta\binom{z}{h}-i$, i.e. $\Theta^*\binom{z-\hat{A}_i^h}{0}\le\Theta\binom{z}{h}-i$. We conclude that $\max_{i=0,\dots,h}\left\{\Theta^*\binom{z-\hat{A}_i^h}{0}+i\right\}\le\Theta\binom{z}{h}$.

Second we prove that $\max_{i=0,\dots,h}\left\{\Theta^*\binom{z-\hat{A}_i^h}{0}+i\right\}\ge\Theta\binom{z}{h}$. Define $M_0=\{j\in\{l_1,\dots,l_h\}:j\ge k_s+1\}$. Consider the solution $\hat{x}$ defined as

$$\hat{x}_j=\begin{cases}x_j^* & \text{if } j\in C\setminus M_0\\ 1 & \text{if } j\in M_0.\end{cases}$$

The solution $\hat{x}$ satisfies

$$\sum_{j\in C}a_j\hat{x}_j=\sum_{j\in C}a_jx_j^*+\sum_{j\in M_0}a_j\le b-z+\hat{A}_{|M_0|}^h$$

and also

$$\sum_{j\in C^+}\hat{x}_j=\sum_{j\in C^+}x_j^*+\sum_{j\in M_0}1\le K-h+|M_0|\le K.$$

It follows that $\hat{x}$ is a feasible solution to the problem $\Theta^*\binom{z-\hat{A}_{|M_0|}^h}{0}$ with objective value $\Theta\binom{z}{h}-|M_0|=h+s-1-|M_0|$. We now prove that $\hat{x}$ is an optimal solution to $\Theta^*\binom{z-\hat{A}_{|M_0|}^h}{0}$. Because $z\in(\hat{A}_h-\lambda+\sum_{j=1}^{s-1}a_{k_j},\hat{A}_h-\lambda+\sum_{j=1}^{s}a_{k_j}]$, we have that

$$z-\hat{A}_{|M_0|}^h\le\hat{A}_h-\lambda+\sum_{j=1}^{s}a_{k_j}-\hat{A}_{|M_0|}^h=\sum_{j=1}^{h-|M_0|}a_{l_j}-\lambda+\sum_{j=1}^{s}a_{k_j}=A_{k_s}-\lambda.$$

The last equality holds because $\{l_1,\dots,l_{h-|M_0|}\}\cup\{k_1,\dots,k_s\}=\{1,\dots,k_s\}$. Similarly, we can show that $z-\hat{A}_{|M_0|}^h>A_{k_s-1}-\lambda$. It follows from (21) that $\Theta\binom{z-\hat{A}_{|M_0|}^h}{0}=\Theta^*\binom{z-\hat{A}_{|M_0|}^h}{0}=k_s-1=h-|M_0|+s-1$. This implies $\Theta^*\binom{z-\hat{A}_{|M_0|}^h}{0}=\Theta\binom{z}{h}-|M_0|$. As a consequence, we have $\max_{i=0,\dots,h}\left\{\Theta^*\binom{z-\hat{A}_i^h}{0}+i\right\}\ge\Theta\binom{z}{h}$. □

4.2. Building a superadditive approximation of Θ

We now construct an approximate lifting function for the generalized cover inequality. First, we give conditions in Corollary 9 that ensure that lifting is sequence-independent. These conditions follow from Theorem 2 and Proposition 3. We define $d^+:=\max\{|S|:S\subseteq(N^+\setminus C^+),\sum_{j\in S}a_j\le b\}$ and denote $K^+:=\min\{K,d^+\}$. We assume without loss of generality that $K^+\ge1$.

Corollary 9. *For PS_1^+, lifting is sequence-independent over $[0,b]\times\{0,1\}$ if*

$$\Theta^*\binom{y}{0}+\Theta^*\binom{z}{0}\le\Theta^*\binom{y+z}{0}\tag{26}$$

$$\Theta^*\binom{y}{0}+\Theta\binom{z}{h}\le\Theta\binom{y+z}{h}\quad\forall h\in\{1,\dots,K^+\}\tag{27}$$

$$\Theta\binom{y}{1}+\Theta\binom{z}{h}\le\Theta\binom{y+z}{h+1}\quad\forall h\in\{1,\dots,K^+-1\}\tag{28}$$

where $y,z,y+z\in[0,b]$. □

Because the conditions of Corollary 9 are most stringent when $K^+=K$, we will construct a superadditive approximation satisfying conditions (26)–(28) for the case where $K^+=K$. Deriving a strong superadditive approximation of an exact lifting function is often cumbersome, even for single-dimensional functions. Therefore, one would expect that building a strong superadditive approximation that satisfies the conditions of Corollary 9 presents a great challenge. However, we will show next that the result of Theorem 8 yields a simple framework to recursively build multi-dimensional superadditive approximations.

In this scheme, the function $\Theta\binom{z}{h}$ is first approximated for $h = 0$. The result of Theorem 8 is then used to extend the approximation to $[0, b] \times \{0, \dots, K\}$. Because $\Theta^*\binom{z}{0}$ is identical to the exact lifting function of the minimal cover inequality, the single-dimensional superadditive approximation of $\Theta^*\binom{z}{0}$ proposed by Gu et al. [5] plays an important role in the derivation of a superadditive approximation for PS_1^+. We present this single-dimensional superadditive approximation next where we define $\rho_i = \max\{0, a_{i+1} - (a_1 - \lambda)\}$ for $i = 0, \dots, |C| - 1$.

Theorem 10 (*Adapted from [5]*). *The function*

$$\theta^*\binom{z}{0} = \begin{cases} 0 & \text{if } z = 0 \\ i & \text{if } A_i - \lambda + \rho_i < z \le A_{i+1} - \lambda \\ i - (A_i - \lambda + \rho_i - z)/\rho_1 & \text{if } A_i - \lambda < z \le A_i - \lambda + \rho_i \end{cases} \tag{29}$$

is a non-dominated approximation of $\Theta^*\binom{z}{0}$ that satisfies $\theta^*\binom{y}{0} + \theta^*\binom{z}{0} \le \theta^*\binom{y+z}{0}$ for $y, z, y + z \in [0, b]$. Furthermore, it is maximal over $[0, b] \times \{0\}$. □

In the remainder of this paper, we define $\theta^*\binom{z}{0} := -\infty$ for $z < 0$. Giving this value to $\theta^*\binom{z}{0}$ where $z < 0$ is not restrictive since all the coefficients of the variables to be lifted are nonnegative. As a direct consequence of Theorems 8 and 10, we easily obtain in Theorem 11 strong multi-dimensional superadditive approximations for $\Theta\binom{z}{h}$ for Cases i and ii of Theorem 8. The derivation for Cases iii and iv is more involved and is presented in Theorem 12.

Theorem 11. *If $C \ne C^+$ and $|C^+| \le K - 1$ or if $C = C^+$ and $|C^+| \le K$, the function $\theta\binom{z}{h} = \theta^*\binom{z}{0}$ for $(z, h) \in [0, b] \times \{0, \dots, K\}$ is a valid superadditive approximation of $\Theta\binom{z}{h}$ that is non-dominated and maximal over $[0, b] \times \{0, \dots, K\}$.*

Proof. From Proposition 1, we conclude that for $z \in [0, b]$, $\Theta\binom{z}{K} \ge \Theta\binom{z}{K-1} \ge \cdots \ge \Theta\binom{z}{1} \ge \Theta^*\binom{z}{0}$. It follows that $\Theta\binom{z}{h} \ge \Theta^*\binom{z}{0} \ge \theta^*\binom{z}{0} = \theta\binom{z}{h}$ for $z \in [0, b]$ and $h \in \{0, \dots, K\}$. Furthermore, for $z_1, z_2 \in [0, b]$ and $h_1, h_2 \in \{0, \dots, K\}$ such that $z_1 + z_2 \in [0, b]$ and $h_1 + h_2 \le K$, we have

$$\theta\binom{z_1}{h_1} + \theta\binom{z_2}{h_2} = \theta^*\binom{z_1}{0} + \theta^*\binom{z_2}{0} \le \theta^*\binom{z_1 + z_2}{0} = \theta\binom{z_1 + z_2}{h_1 + h_2},$$

showing that the conditions of Corollary 9 are satisfied. The non-dominance and maximality of θ follow directly from Theorems 8 and 10 since $\Theta\binom{z}{1} = \Theta^*\binom{z}{0}$. □

Clearly, because $\theta_0(z - a_1) + 1 \le \theta_0(z)$, θ defined in Theorem 11 has the form described in Theorem 6 if we set $a^* := a_1$. The result of Theorem 11 illustrates the fact that, in this case, multi-dimensional lifting is not stronger than one-dimensional lifting. However, in general, we can obtain stronger cuts using high-dimensional superadditive approximations that consider multiple constraints simultaneously. Next, in Theorem 12, we propose a valid multi-dimensional superadditive approximation for Θ in Cases iii and iv of Theorem 8 that yields cutting planes stronger than those obtained from one-dimensional superadditive lifting.

Theorem 12. *Assume that $C \ne C^+$ and $|C^+| = K$ or $C = C^+$ and $|C^+| = K + 1$. The function $\theta\binom{z}{h}$ defined as*

$$\theta\binom{z}{h} = \begin{cases} \theta^*\binom{z}{0} & \text{if } h = 0 \\ \begin{cases} \theta^*\binom{z}{0} & \text{if } \quad 0 \le z < p \\ \max\{\theta^*\binom{z}{0}, \theta^*\binom{z-p}{0} + 1\} & \text{if } \quad p \le z \le b \end{cases} & \text{if } h = 1 \\ \sup\limits_{\left\{z = \sum_{j=1}^{h} z_j : z_j \ge 0, j = 1, \dots, h\right\}} \sum_{j=1}^{h} \theta\binom{z_j}{1} & \text{if } h = 2, \dots, K \end{cases} \tag{30}$$

where

$$p = \begin{cases} a_{l_1} & \text{when } C \ne C^+ \text{ and } |C^+| = K \\ a_2 & \text{when } C = C^+ \text{ and } |C^+| = K + 1 \end{cases} \tag{31}$$

is a valid superadditive approximation of the exact lifting function $\Theta\binom{z}{h}$ for $(z, h) \in [0, b] \times \{0, \dots, K\}$.

Proof. We only present the proof for Case iii where $C \ne C^+$ and $|C^+| = K$ since the result for Case iv where $C = C^+$ and $|C^+| = K + 1$ can be shown using a similar argument. The proof is organized in three steps. First we show that $\theta\binom{z}{h} \le \Theta\binom{z}{h}$ for $(z, h) \in [0, b] \times \{0, 1\}$. Second, we prove that the functions $\theta\binom{z}{0}$ and $\theta\binom{z}{1}$ satisfy conditions (26) and (27) of Corollary 9.

Note that because of the way we build $\theta\binom{z}{h}$ for $h \geq 2$, the remaining superadditive conditions in Corollary 9 are naturally satisfied. Third, we show that $\theta\binom{z}{h}$ is a valid lower approximation for $\Theta\binom{z}{h}$ for $(z, h) \in [0, b] \times \{2, \dots, K\}$.

Since $\theta^*\binom{z}{0}$ is the valid superadditive approximation of $\Theta^*\binom{z}{0}$ proposed by Gu et al. [5], $\theta\binom{z}{0} = \theta^*\binom{z}{0} \leq \Theta^*\binom{z}{0}$ for $z \in [0, b]$. Now, for $z \in [0, a_{l_1})$, we deduce from Proposition 1 that $\Theta\binom{z}{1} \geq \Theta^*\binom{z}{0} \geq \theta\binom{z}{1} = \theta^*\binom{z}{0}$. For $z \in [a_{l_1}, b]$, we conclude from Theorem 8 that

$$\Theta\binom{z}{1} = \max\left\{\Theta^*\binom{z-a_{l_1}}{0} + 1, \Theta^*\binom{z}{0}\right\} \geq \max\left\{\theta^*\binom{z-a_{l_1}}{0} + 1, \theta^*\binom{z}{0}\right\} = \theta\binom{z}{1}.$$

We now verify that θ satisfies the conditions (26) and (27). The fact that condition (26) is satisfied follows from [5]. To verify condition (27), we must show that

$$\theta\binom{y}{0} + \theta\binom{z}{1} \leq \theta\binom{y+z}{1}$$

for $y, z, y+z \in [0, b]$. For $y \in [0, b]$, $z \in [0, a_{l_1})$ and $y + z \in [0, b]$, we have

$$\theta\binom{y}{0} + \theta\binom{z}{1} = \theta^*\binom{y}{0} + \theta^*\binom{z}{0} \leq \theta^*\binom{y+z}{0} \leq \theta\binom{y+z}{1}$$

since it is easily verified that $\theta\binom{u}{0} \leq \theta\binom{u}{1}$ for $u \in [0, b]$. For $y \in [0, b]$, $z \in [a_{l_1}, b]$ and $y + z \in [0, b]$, we have

$$\begin{aligned}\theta\binom{y}{0} + \theta\binom{z}{1} &= \max\left\{\theta^*\binom{y}{0} + \theta^*\binom{z-a_{l_1}}{0} + 1, \theta^*\binom{y}{0} + \theta^*\binom{z}{0}\right\}\\ &\leq \max\left\{\theta^*\binom{y+z-a_{l_1}}{0} + 1, \theta^*\binom{y+z}{0}\right\} = \theta\binom{y+z}{1}\end{aligned}$$

since $y + z \geq a_{l_1}$.

Finally, using the fact that $\theta\binom{z}{h}$ is a valid lower superadditive approximation of $\Theta\binom{z}{h}$ for $h \in \{0, 1\}$, we now prove inductively that $\theta\binom{z}{h}$ is a valid lower approximation for $h \geq 2$. Assume that we have already proven that $\theta\binom{z}{h} \leq \Theta\binom{z}{h}$ for $z \in [0, b]$ and $h = 1, \dots, t$. We want to show that this result still holds for $h = t + 1$. For $z \in [0, b]$, we define

$$R_1 := \sup_{\left\{z=\sum_{j=1}^{t+1} z_j : z_j \geq 0, j=1,\dots,t, z_{t+1} \in [0, a_{l_1})\right\}} \sum_{j=1}^{t+1} \theta\binom{z_j}{1} \tag{32}$$

and

$$R_2 := \sup_{\left\{z=\sum_{j=1}^{t+1} z_j : z_j \geq a_{l_1}, j=1,\dots,t+1\right\}} \sum_{j=1}^{t+1} \theta\binom{z_j}{1}. \tag{33}$$

Clearly, $\theta\binom{z}{t+1} \leq \max\{R_1, R_2\}$. We now prove that $R_1 \leq \Theta\binom{z}{t+1}$ and $R_2 \leq \Theta\binom{z}{t+1}$. For R_1, we observe first that $\theta\binom{z_{t+1}}{1} = \theta\binom{z_{t+1}}{0}$ since $z_{t+1} \in [0, a_{l_1})$. We have

$$\begin{aligned}R_1 &\leq \sup_{\left\{z=\sum_{j=1}^{t+1} z_j : z_j \geq 0, j=1,\dots,t, z_{t+1} \in [0, a_{l_1})\right\}} \left\{\sum_{j=1}^{t-1} \theta\binom{z_j}{1} + \theta\binom{z_t + z_{t+1}}{1}\right\}\\ &\leq \sup_{\left\{z=\sum_{j=1}^{t} \tilde{z}_j : \tilde{z}_j \geq 0, j=1,\dots,t\right\}} \left\{\sum_{j=1}^{t} \theta\binom{\tilde{z}_j}{1}\right\}\\ &= \theta\binom{z}{t} \leq \Theta\binom{z}{t} \leq \Theta\binom{z}{t+1}\end{aligned}$$

where the first inequality holds because of the superadditivity of $\theta\binom{z}{h}$ over $[0, b] \times \{0, 1\}$. Next, we prove that $R_2 \leq \Theta\binom{z}{t+1}$. Let $z_j \geq a_{l_1}$ for $j = 1, \dots, t + 1$ be such that $\sum_{j=1}^{t+1} z_j = z$. By (30), we can reorder z_js to have

$$\sum_{j=1}^{t+1} \theta\binom{z_j}{1} = \sum_{j=1}^{k(z)} \left(\theta\binom{z_j - a_{l_1}}{0} + 1\right) + \sum_{j=k(z)+1}^{t+1} \theta\binom{z_j}{0}$$

where $0 \le k(z) \le t+1$. When $k(z) = 0$, it is clear that $R_2 \le \Theta\binom{z}{t+1}$ since that $\theta^*\binom{z}{0}$ is a valid superadditive approximation of $\Theta^*\binom{z}{0}$ and $\Theta^*\binom{z}{0} \le \Theta\binom{z}{t+1}$. When $1 \le k(z) \le t$, we have

$$\begin{aligned}\sum_{j=1}^{t+1}\theta\binom{z_j}{1} &\le \sum_{j=1}^{k(z)-1}\left(\theta\binom{z_j - a_{l_1}}{0} + 1\right) + \theta\begin{pmatrix} z_{k(z)} + \sum_{j=k(z)+1}^{t+1} z_j - a_{l_1} \\ 0\end{pmatrix} + 1 \\ &\le \sum_{j=1}^{k(z)-1}\theta\binom{z_j}{1} + \theta\begin{pmatrix} z_{k(z)} + \sum_{j=k(z)+1}^{t+1} z_j \\ 1\end{pmatrix} \\ &\le \theta\binom{z}{k(z)} \le \Theta\binom{z}{k(z)} \le \Theta\binom{z}{t+1}. \end{aligned} \tag{34}$$

When $k(z) = t+1$, because $\hat{A}_{t+1}^{t+1} \le (t+1)a_{l_1}$ and because of Theorem 8, we have

$$\begin{aligned}\sum_{j=1}^{t+1}\theta\binom{z_j}{1} &\le \theta\binom{z-(t+1)a_{l_1}}{0} + t + 1 \\ &\le \Theta^*\binom{z - \hat{A}_{t+1}^{t+1}}{0} + t + 1 \\ &\le \Theta\binom{z}{t+1}. \end{aligned} \tag{35}$$

Therefore, we conclude that $R_2 = \sup_{\{z=\sum_{j=1}^{t+1} z_j : z_j \ge a_{l_1}, j=1,\dots,t+1\}} \sum_{j=1}^{t+1}\theta\binom{z_j}{1} \le \Theta\binom{z}{t+1}$. □

Because $\theta\binom{z}{0} = -\infty$ when $z < 0$, we see that the result of Theorem 12 corresponds to that of Theorem 6 with $a^* := a_{l_1}$. In the case where $C = C^+$ and $|C^+| = K+1$, it is also easy to show that it corresponds to Theorem 6 if we set $a^* = a_2$.

Next, we prove that the superadditive function θ we derived in Theorem 12 is a strong approximation of Θ. We emphasize that it is only necessary to show the strength of θ over $[0, b] \times \{0, 1\}$, the region that the coefficients of the variables to be lifted belong to.

Theorem 13. *Assume that $C \ne C^+$ and $|C^+| = K$ or $C = C^+$ and $|C^+| = K+1$. The function $\theta\binom{z}{h}$ defined in Theorem 12 is a non-dominated and maximal superadditive approximation of $\Theta\binom{z}{h}$ over $[0, b] \times \{0, 1\}$.*

Proof. Because the proofs are very similar, we only show the result for Case iii where $C \ne C^+$ and $|C^+| = K$. First, we give the following explicit form for $\Theta\binom{z}{1}$:

$$\Theta\binom{z}{1} = \begin{cases} 0 & \text{if } 0 \le z \le a_{l_1} - \lambda \\ i & \text{if } A_{i-1} + a_{l_1} - \lambda < z \le A_i + a_{l_1} - \lambda \; i = 1, \dots, l_1 - 1 \\ i & \text{if } A_i - \lambda < z \le A_{i+1} - \lambda \; i = l_1, \dots, |C| - 1. \end{cases} \tag{36}$$

We now prove that $\theta\binom{z}{h}$ is non-dominated. When $a_{l_1} = a_1$, $\Theta\binom{z}{1} = \Theta^*\binom{z}{0}$ and $\theta\binom{z}{1} = \theta^*\binom{z}{0}$, the result then follows from Theorem 10. It is therefore sufficient to consider the case where $a_{l_1} < a_1$. Assume by contradiction that θ is dominated, i.e. there exists a superadditive function $\theta' : [0, b] \times \{0, 1\} \longmapsto \mathbb{R}$ such that $\theta'\binom{z}{h} \ge \theta\binom{z}{h}$ for $(z, h) \in [0, b] \times \{0, 1\}$ and $\theta'\binom{z'}{h'} > \theta\binom{z'}{h'}$ for some $(z', h') \in [0, b] \times \{0, 1\}$. It follows from Theorem 10 that $h' = 1$. Further, note that because $a_i \ge a_{l_1}$ for all $i \in C$ such that $i < l_1$, we have $\Theta^*\binom{z-a_{l_1}}{0} + 1 \ge \Theta^*\binom{z}{0}$ for $a_{l_1} \le z \le A_{l_1} - \lambda$ and because $a_j \le a_{l_1}$ for all $j \in C$ such that $j > l_1$, we have $\Theta^*\binom{z-a_{l_1}}{0} + 1 \le \Theta^*\binom{z}{0}$ for $A_{l_1} - \lambda < z \le b$. Similarly, it is easy to verify that

$$\theta\binom{z}{1} = \begin{cases} \theta^*\binom{z}{0} & \text{if } 0 \le z < a_{l_1} \\ \theta^*\binom{z - a_{l_1}}{0} + 1 & \text{if } a_{l_1} \le z \le A_{l_1} - \lambda \\ \theta^*\binom{z}{0} & \text{if } A_{l_1} - \lambda < z \le b. \end{cases} \tag{37}$$

We now consider three cases.

(1) If $z' \in [0, a_{l_1})$, then $0 \le \theta\binom{z'}{1} < \theta'\binom{z'}{1} \le \Theta\binom{z'}{1}$. Because $\Theta\binom{z}{1} = 0$ when $z \in [0, a_{l_1} - \lambda]$, we conclude that $z' \in (a_{l_1} - \lambda, a_{l_1})$ with $\Theta\binom{z'}{1} = 1$ and $\theta\binom{z'}{1} = \theta^*\binom{z'}{0} < 1$. Now consider $z = A_{l_1} - \lambda$. Clearly, $z - z' \in (A_{l_1-1} - \lambda, A_{l_1-1}]$ since $z' < a_{l_1}$. It follows from Theorems 7 and 10 that $\Theta\binom{z-z'}{0} = l_1 - 1$ and $\theta^*\binom{z-z'}{0} \in (l_1 - 2, l_1 - 1]$. On one hand, if $\theta^*\binom{z-z'}{0} = l_1 - 1$, then $\theta'\binom{z-z'}{0} \ge l_1 - 1 = \Theta\binom{z-z'}{1} \ge \Theta^*\binom{z-z'}{0}$. Therefore, $\theta'\binom{z-z'}{0} = l_1 - 1$. Furthermore,

$$l_1 - 1 + \theta'\binom{z'}{1} = \theta'\binom{z-z'}{0} + \theta'\binom{z'}{1} \le \theta'\binom{z}{1} \le \Theta\binom{z}{1} = l_1 - 1.$$

It follows that $\theta'\binom{z'}{1} \le 0$, which is the desired contradiction.

On the other hand, if $l_1 - 2 < \theta^*\binom{z-z'}{0} < l_1 - 1$, then it follows from Theorem 10 that $A_{l_1-1} - \lambda < z - z' \le A_{l_1-1} - \lambda + \rho_{l_1-1}$ with $\rho_{l_1-1} = a_{l_1} - (a_1 - \lambda) > 0$. It follows that $z' > a_{l_1} - \rho_{l_1-1} = a_1 - \lambda$. Furthermore,

$$\begin{aligned} l_1 - 1 = \Theta\binom{z}{1} &\ge \theta'\binom{z}{1} \ge \theta'\binom{z'}{1} + \theta'\binom{z-z'}{0} > \theta\binom{z'}{1} + \theta^*\binom{z-z'}{0} \\ &= \frac{z' - a_1 + \lambda}{\rho_1} + l_1 - 1 - \frac{\rho_{l_1-1} - a_{l_1} + z'}{\rho_1} = l_1 - 1 \end{aligned}$$

where the next to the last inequality holds because of (29) and (37). This is the desired contradiction.

(2) If $z' \in (a_{l_1}, A_{l_1} - \lambda]$, then using (37), $\theta\binom{z'}{1} = \theta^*\binom{z'-a_{l_1}}{0} + 1$ and $\Theta\binom{z'}{1} = \Theta^*\binom{z'-a_{l_1}}{0} + 1$. Also, because $\theta\binom{z'}{1} < \theta'\binom{z'}{1} \le \Theta\binom{z'}{1}$, it follows that $z' - a_{l_1} \in (A_{j-1} - \lambda, A_{j-1} - \lambda + \rho_{j-1})$ for some $j \in \{2, \dots, l_1\}$ such that $\rho_{j-1} > 0$. Define now $z = A_j + a_{l_1} - \lambda$. From (36) and (37), $\Theta\binom{z}{1} = \theta\binom{z}{1} = j$. Clearly, $z - z' \in (a_j - \rho_{j-1}, a_j) \subseteq (a_1 - \lambda, a_j)$. From (30), we obtain

$$j \ge \theta'\binom{z}{1} \ge \theta'\binom{z'}{1} + \theta'\binom{z-z'}{0} > \theta\binom{z'}{1} + \theta^*\binom{z-z'}{0} = \theta^*\binom{z'-a_{l_1}}{0} + 1 + \theta^*\binom{z-z'}{0} = j,$$

which is the desired contradiction.

(3) If $z' \in (A_{l_1} - \lambda, b]$, then $\theta\binom{z'}{1} = \theta^*\binom{z'}{0}$ and $\Theta\binom{z'}{1} = \Theta^*\binom{z'}{0}$. This is a contradiction to the fact that $\theta^*\binom{z}{0}$ is non-dominated over $[0, b]$.

Finally, we prove that $\theta\binom{z}{h}$ is a maximal approximation of $\Theta\binom{z}{h}$ over $[0, b] \times \{0, 1\}$. Let $E \subseteq [0, b] \times \{0, 1\}$ be the maximal set of $\Theta\binom{z}{h}$. We show that if $\theta\binom{z'}{h'} < \Theta\binom{z'}{h'}$ for some $(z', h') \in [0, b] \times \{0, 1\}$, then $(z', h') \notin E$. When $h' = 0$, the proof reduces to that of Theorem 10; see [5]. Assume therefore that $h' = 1$. We consider the following three cases.

(1) If $z' \in [0, a_{l_1})$, we must have $z' \in (a_{l_1} - \lambda, a_{l_1})$ since $\theta\binom{u}{1} = \Theta\binom{u}{1}$ for $u \in [0, a_{l_1} - \lambda]$. Let $z = A_{l_1} - \lambda$. Then $\Theta\binom{z}{1} = l_1 - 1$. Note also that $A_{l_1-1} - \lambda < z - z' \le A_{l_1} - \lambda$ and so $\Theta^*\binom{z-z'}{0} = l_1 - 1$. During the sequential lifting procedure, we have

$$\Theta_2\binom{z'}{1} = \min\left\{\Theta\binom{z'}{1}, \Theta\binom{z}{1} - \Theta^*\binom{z-z'}{0}\right\} = \min\{1, 0\} = 0.$$

We conclude that $(z', 1) \notin E$.

(2) If $z' \in [a_{l_1}, A_{l_1} - \lambda]$, we must have $z' - a_{l_1} \in (A_{j-1} - \lambda, A_{j-1} - \lambda + \rho_{j-1})$ for some $j \in \{2, \dots, l_1\}$ with $\rho_{j-1} > 0$. Consider $z = A_j + a_{l_1} - \lambda$. Then, $\Theta\binom{z'}{1} = \Theta\binom{z}{1} = j$ and $z - z' > a_j - \rho_{j-1} = a_1 - \lambda$. We conclude that $\Theta^*\binom{z-z'}{0} \ge 1$. Furthermore, during the sequential lifting procedure, we have

$$\Theta_2\binom{z'}{1} = \min\left\{\Theta\binom{z'}{1}, \Theta\binom{z}{1} - \Theta^*\binom{z-z'}{0}\right\} \le \min\{j, j-1\} = j - 1,$$

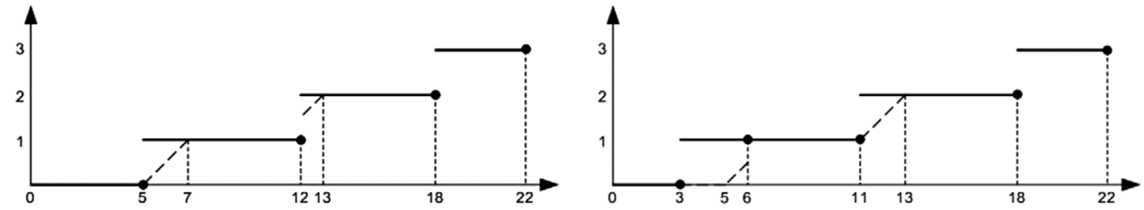
showing that $(z', 1) \notin E$.

(3) If $z' \in (A_{l_1} - \lambda, b]$, then $\theta\binom{z'}{1} = \theta^*\binom{z'}{0}$ and $\Theta\binom{z'}{1} = \Theta^*\binom{z'}{0}$. Therefore, the proof reduces to that of Theorem 10. □

We next illustrate the strength of lifted generalized cover inequalities on an example. This example is obtained from [5] page 122 by adding a cardinality constraint.

Example 2. Consider

$$PS_1 = \operatorname{conv}\left\{x \in \{0, 1\}^5 : 8x_1 + 7x_2 + 6x_3 + 4x_4 + 6x_5 \le 22,\ x_3 + x_4 + x_5 \le 2\right\}.$$



(a) $\Theta^*\binom{z}{0}$ and $\theta\binom{z}{0}$. (b) $\Theta\binom{z}{1}$ and $\theta\binom{z}{1}$.

Fig. 1. Exact and superadditive lifting functions for Example 2.

Clearly, $C = \{1, 2, 3, 4\}$ is a generalized cover with $\lambda = \sum_{j=1}^{4} a_j - 22 = 3$ and $a_{l_1} = 6$. Fig. 1 shows both the exact lifting function Θ and its superadditive approximation θ.

Denote the lifting coefficient of x_5 by α_5. Using the traditional superadditive lifting function from [5], we obtain $\alpha_5 = \theta^*\begin{pmatrix}6\\0\end{pmatrix} = \frac{1}{2}$. Using our two-dimensional superadditive approximation, we obtain $\alpha_5 = \theta\begin{pmatrix}6\\1\end{pmatrix} = 1$. Furthermore, we observe that the inequality obtained using multi-dimensional lifting is facet-defining for PS_1. □

5. Sequence-independent lifting in PS_r^+

In this section, we study sequence-independent lifting for generalized cover inequalities in PS_r^+. As we discussed earlier, because instances of PS_r can be relaxed into instances of PS_r^+, the results derived in this section can also be used to generate valid inequalities for PS_r.

In the remainder of this section, we use Ω to denote the exact lifting function of the generalized cover inequality in PS_r^+ and use ω to denote its superadditive approximation. We let $\vec{I}_j = \{I_{j1}, \ldots, I_{jr}\}^T \in \mathbb{R}^r$ be the vector of the coefficients of x_j in the r cardinality constraints. Because we assume that the cardinality constraints are disjoint, $\vec{I}_j$ can be either $\vec{0}$ or one of the unit vectors e_i of $\mathbb{R}^r$. Recall that a generalized cover C in PS_r^+ is a minimal cover for the knapsack constraint.

The following theorem describes the main result that is proven in this section.

Theorem 14. *Let C be a generalized cover satisfying $C \not\subseteq N_i$ for all $i \in R \cup \{0\}$. Then the inequality*

$$\sum_{j\in C} x_j + \sum_{j\in N\setminus C} \omega\begin{pmatrix}a_j\\ \vec{I}_j\end{pmatrix} x_j \le |C| - 1 \tag{38}$$

is valid and maximal for PS_r^+ when

$$\omega\begin{pmatrix}z\\ \vec{I}_j\end{pmatrix} = \begin{cases}\theta^*\begin{pmatrix}z\\0\end{pmatrix} & \text{if } \vec{I}_j = \vec{0}\text{; or if } \vec{I}_j = e_i \text{ and } \eta_i \ge 1\\ \max\left\{\theta^*\begin{pmatrix}z - a_i^*\\0\end{pmatrix} + 1, \theta^*\begin{pmatrix}z\\0\end{pmatrix}\right\} & \text{if } \vec{I}_j = e_i \text{ and } \eta_i = 0\end{cases} \tag{39}$$

with $a_i^* = \max\{a_j : j \in C_i\}$. □

Again, we only present the superadditive approximation ω of Ω over $\{\vec{0}, e_1, \ldots, e_r\}$ in Theorem 14 to streamline the exposition. The complete superadditive approximation is presented in Proposition 17. It can be verified that, in general, (38) dominates the cuts generated from the knapsack constraint only using $\theta^*\binom{z}{0}$.

5.1. Exact lifting function of the generalized cover inequality

We first give a description of the tth lifting problem of the generalized cover inequality

$$\begin{aligned}\Omega_t\begin{pmatrix}z\\ \vec{v}\end{pmatrix} = \min\ & |C| - 1 - \sum_{j\in C} x_j - \sum_{1\le s<t} \alpha_s x_s\\ \text{s.t. } & \sum_{j\in C} a_j x_j + \sum_{1\le s<t} a_s x_s \le b - z\\ & \sum_{j\in C_i} x_j + \sum_{1\le s<t} I_{si} x_s \le K_i - v_i, \quad i \in R\\ & x_j \in \{0, 1\} \quad \forall j \in C \cup \{1, \ldots, t-1\}\end{aligned} \tag{40}$$

where $\alpha_s = \Omega_s\begin{pmatrix}a_s\\ \vec{I}_s\end{pmatrix}$, $z \in [0, b]$ and $\vec{v} = \{v_1, \ldots, v_r\}^T \in \mathbb{Z}_+^r$. We denote Ω_1 by Ω.

For the special case where $C \subseteq N_k$ for some $k \in R$, it is easy to see that the lifting function $\Omega\binom{z}{e_i} = \Omega\binom{z}{\vec{0}} = \Theta^*\binom{z}{0}$ for $i \neq k$. Therefore, we can build an approximate lifting function for $\Omega\binom{z}{\vec{v}}$ by using the closed-form approximation of $\Theta\binom{z}{0}$ for $\Omega\binom{z}{e_i}$ when $i \neq k$ and by using the approximation derived for PS_1^+ where $C \subseteq N_1$ for $\Omega\binom{z}{e_k}$. A similar argument applies to the case where $C \subseteq N_0$. Therefore, unless otherwise mentioned, we assume in the remainder of this section that $C \nsubseteq N_i$ for $i \in R \cup \{0\}$.

We define $R^{=} = \{i \in R : \eta_i = 0\}$ to represent the set composed of all cardinality constraints i whose intersections with the generalized cover have exactly K_i variables. We define $\overline{R^{=}}$ as its complement. We let $C_i = \{j_{i,1}, \ldots, j_{i,K_i-\eta_i}\}$ for $i \in R$ and assume without loss of generality that $a_{j_{i,1}} \geq \cdots \geq a_{j_{i,K_i-\eta_i}}$. Similar to the discussion for PS_1^+, we define $A_{i,k} = \sum_{s=1}^{k} a_{j_{i,s}}$ and $\hat{A}_{i,k}^{v} = \sum_{s=v-k+1}^{v} a_{j_{i,s}}$ with $k = 1, \ldots, v$ and $v = 1, \ldots, K_i - \eta_i$ for all $i \in R$. We also let $d_i^+ = \max\left\{|S| : S \subseteq N_i \setminus C_i, \sum_{k \in S} a_k \leq b\right\}$ and $K_i^+ = \min\{d_i^+, K_i\}$ for $i \in R$. We define $\mathbb{D} = \{0, \ldots, K_1^+\} \times \cdots \times \{0, \ldots, K_r^+\}$ and $\mathbb{D}' = \{\vec{0}, e_1, \ldots, e_r\}$. In addition, for $\vec{v} \in \mathbb{D}$, we introduce the notation $v_i^+ = \max\{v_i - \eta_i, 0\}$ for $i \in R$ to represent the minimal number of elements of the cover to be removed from C_i in any feasible solution to $\Omega\binom{z}{\vec{v}}$. Finally, we define $\vec{v}^+ = \{v_1^+, \ldots, v_r^+\}$.

From (40), it is easy to verify that $\Omega\binom{z}{\vec{0}} = \Theta^*\binom{z}{0}$. Using an argument similar to that of Theorem 8, we obtain the form for $\Omega\binom{z}{\vec{v}}$ that is described in Theorem 15.

Theorem 15. *The exact lifting function of the generalized cover inequality is*

$$\Omega\binom{z}{\vec{v}} = \begin{cases} \Theta^*\binom{z}{0} & \text{if} \quad \sum_{i\in R} v_i^+ = 0 \\ \begin{cases} \sum_{i\in R} v_i^+ - 1 & \text{if} \quad 0 \leq z \leq \sum_{i\in R} A_{i,v_i^+} - \lambda \\ \sum_{i\in R} v_i^+ & \text{if} \quad \sum_{i\in R} A_{i,v_i^+} - \lambda < z < \sum_{i\in R} A_{i,v_i^+} \\ \hat{\Omega}\binom{z}{\vec{v}} & \text{if} \quad \sum_{i\in R} A_{i,v_i^+} \leq z \leq b \end{cases} & \text{if} \quad \sum_{i\in R} v_i^+ \geq 1 \end{cases} \tag{41}$$

for $(z, \vec{v}) \in [0, b] \times \mathbb{D}$ *where* $\hat{\Omega}\binom{z}{\vec{v}} = \max_{k_i:0\leq k_i\leq v_i^+, i\in R}\left\{\Theta^*\begin{pmatrix} z-\sum_{i\in R}\hat{A}_{i,k_i}^{v_i^+} \\ \vec{0}\end{pmatrix} + \sum_{i\in R} k_i\right\}$. $\square$

From (41), it can be verified that $\Omega\binom{z}{e_i} = \Theta^*\binom{z}{0}$ if $i \in \overline{R^{=}}$. If $i \in R^{=}$, $\Omega\binom{z}{e_i}$ has the form of $\Theta\binom{z}{1}$ defined in (24) where a_{l_s} is replaced with $a_{j_{i,s}}$.

5.2. Building a superadditive approximation of Ω

Similar to our discussion on PS_1^+, we consider in this section the most stringent case $\mathbb{D} = \{0, \ldots, K_1\} \times \cdots \times \{0, \ldots, K_r\}$. We derive next in Corollary 16 sufficient conditions for sequence-independent lifting. This corollary follows from Theorem 2 and Proposition 3.

Corollary 16. *For* PS_r^+, *lifting is sequence-independent if*

$$\Omega\binom{y}{\vec{0}} + \Omega\binom{z}{\vec{0}} \leq \Omega\binom{y+z}{\vec{0}} \tag{42}$$

$$\Omega\binom{y}{\vec{0}} + \Omega\binom{z}{e_i} \leq \Omega\binom{y+z}{e_i} \tag{43}$$

$$\Omega\binom{y}{\vec{0}} + \Omega\binom{z}{\vec{v}} \leq \Omega\binom{y+z}{\vec{v}} \tag{44}$$

$$\Omega\binom{y}{e_i} + \Omega\binom{z}{\vec{v}} \leq \Omega\binom{y+z}{e_i+\vec{v}} \tag{45}$$

for all $y, z, y+z \in [0, b]$, $i \in R$ *and* $\vec{v} \in \mathbb{D}$. $\square$

Note that conditions (42) and (43) are implied by (44) and (45). However, we list them separately in Corollary 16 because they are central to the derivation of the superadditive approximation. Now, using a construction similar to that of Theorem 12, we give a valid superadditive approximation of Ω over $[0, b] \times \mathbb{D}$.

Proposition 17. *The function*

$$\omega\begin{pmatrix} z \\ \vec{v} \end{pmatrix} = \begin{cases} \theta^*\begin{pmatrix} z \\ 0 \end{pmatrix}, & \text{if } \vec{v}^+ = \vec{0} \\ \max\{\theta^*\begin{pmatrix} z - a_{j_{i,1}} \\ 0 \end{pmatrix} + 1, \theta^*\begin{pmatrix} z \\ 0 \end{pmatrix}\}, & \text{if } \vec{v}^+ = e_i \\ \sup\limits_{\left\{z = \sum_{i\in R^=, 1\le j\le v_i^+} z_{i,j},\ z_{i,j}\ge 0\ \forall i,j\right\}} \left\{ \sum\limits_{i\in R^=, 1\le j\le v_i^+} \omega\begin{pmatrix} z_{i,j} \\ e_i \end{pmatrix} \right\} & \text{if } \vec{v}^+ \notin \mathbb{D}' \end{cases} \tag{46}$$

for $\vec{v} \in \mathbb{D}$ and $i \in R$ is a valid superadditive approximation of $\Omega\binom{z}{\vec{v}}$ for $(z, \vec{v}) \in [0, b] \times \mathbb{D}$. Furthermore, it is maximal and non-dominated over $(z, \vec{v}) \in [0, b] \times \mathbb{D}'$.

Proof. The validity and superadditivity of ω follow directly from Theorems 12 and 15 and the way we build $\omega\binom{z}{\vec{v}}$ if $\vec{v}^+ \notin \mathbb{D}'$. Also, because the non-dominance is easy to verify, we only give the proof that ω is maximal. Let $E^r \subseteq [0, b] \times \mathbb{D}'$ be the maximal set of (8) for PS_r^+. It is sufficient to show that if $\omega\binom{z_0}{\vec{v}_0} < \Omega\binom{z_0}{\vec{v}_0}$ for $(z_0, \vec{v}_0) \in [0, b] \times \mathbb{D}'$, then $(z_0, \vec{v}_0) \notin E^r$.

Consider first the case where $\vec{v}_0 = \vec{0}$. Because $\omega\binom{z_0}{\vec{0}} = \theta^*\binom{z_0}{0}$ and $\Omega\binom{z_0}{\vec{0}} = \Theta^*\binom{z_0}{0}$, the proof that $\Omega_2\binom{z_0}{\vec{0}} \le \min\left\{\Omega\binom{z_0}{\vec{0}}, \Omega\binom{z_0+y}{\vec{0}} - \Omega\binom{y}{\vec{0}}\right\} < \Omega\binom{z_0}{\vec{0}}$ for some y such that $y, z_0 + y \in [0, b]$ reduces to that of Theorem 10.

Consider now the case where $\vec{v}_0 = e_i$ for some $i \in R$. It is sufficient to consider $i \in R^=$ since $\Omega\binom{z}{e_i} = \Omega\binom{z}{\vec{0}}$ when $i \in \overline{R^=}$. It follows from Proposition 17 and from the proof of Theorem 13 that there are two situations: (i) $\omega\binom{z_0}{e_i} = \theta^*\binom{z_0}{0}$ and $\Omega\binom{z_0}{e_i} = \Theta^*\binom{z_0}{0}$, or (ii) $\omega\binom{z_0}{e_i} = \theta^*\binom{z_0 - a_{j_{i,1}}}{0} + 1$ and $\Omega\binom{z_0}{e_i} = \Theta^*\binom{z_0 - a_{j_{i,1}}}{0} + 1$. The proofs of these cases reduces to those given in Theorem 13. Therefore, we conclude that $(z_0, e_i) \notin E^r$. □

Next, we present an example from [11] that illustrates the strength of our multi-dimensional superadditive approximation.

Example 3 (*[11]*)**.** Consider

$$PS_2 = \operatorname{conv}\big\{x \in \{0, 1\}^{10}: 37x_1 + 25x_2 + 23x_3 + 15x_4 + 14x_5 + 12x_6 + 11x_7 + 8x_8 + 7x_9 + 3x_{10} \le 39,$$
$$x_6 + x_7 \le 1, x_9 + x_{10} \le 1\big\}.$$

It is easy to verify that $C = \{4, 5, 8, 10\}$ is a generalized cover with $\lambda = 1$, $\eta_1 = 1$ and $\eta_2 = 0$. Therefore, from Theorem 15, we have

$$\Omega\begin{pmatrix} z \\ \vec{0} \end{pmatrix} = \Omega\begin{pmatrix} z \\ e_1 \end{pmatrix} = \begin{cases} 0 & \text{if } 0 \le z \le 14 \\ 1 & \text{if } 14 < z \le 28 \\ 2 & \text{if } 28 < z \le 36 \\ 3 & \text{if } 36 < z \le 39, \end{cases}$$

and

$$\Omega\begin{pmatrix} z \\ e_2 \end{pmatrix} = \begin{cases} 0 & \text{if } 0 \le z \le 2 \\ 1 & \text{if } 2 < z \le 17 \\ 2 & \text{if } 17 < z \le 31 \\ 3 & \text{if } 31 < z \le 39. \end{cases}$$

From Theorem 10 and Proposition 17, it is easily seen that $\omega\binom{z}{\vec{0}} = \omega\binom{z}{e_1} = \Omega\binom{z}{\vec{0}} = \Omega\binom{z}{e_1}$ and $\omega\binom{z}{e_2} = \max\left\{\omega\binom{z-3}{\vec{0}} + 1, \omega\binom{z}{\vec{0}}\right\}$. Using this superadditive approximation, we derive

$$3x_1 + x_2 + x_3 + x_4 + x_5 + x_8 + x_9 + x_{10} \le 3. \tag{47}$$

This inequality is proven to be facet-defining for PS_2 in [11]. □

6. Computational experiments

In this section, we report preliminary computational results on the effectiveness of cutting planes generated using the sequence-independent lifting method introduced in this paper and using the sequential lifting method presented in [1].

Because our superadditive approximate lifting functions are designed for PS_r^+, we only perform our experiments on 0–1 integer programs with positive coefficients. The problems we study are of the form

$$\begin{aligned} \min \quad & \sum_{j\in N} c_j x_j \\ \text{s.t.} \quad & \sum_{j\in N} a_{i,j} x_j \le b_i, \quad \forall i \in M \\ & \sum_{j\in N_l} x_j \le K_l, \quad \forall l \in CAR \\ & x_j \in \{0,1\}, \quad \forall j \in N \end{aligned} \tag{48}$$

where M denotes the set of knapsack constraints and CAR denotes the set of cardinality constraints. We set $|N| = 600$, $|M| = 6$ and $|CAR| = 6$. The coefficients $a_{i,j}$ and b_i for all i, j are integers generated uniformly at random from intervals $[200, 800]$ and $[20\,000, 70\,000]$ respectively. Coefficients c_j are uniform random integers drawn from the interval $\left[\alpha_1 \left(\sum_{i\in M} a_j + \beta|CAR|\right), \alpha_2 \left(\sum_{i\in M} a_j + \beta|CAR|\right)\right]$ with $\alpha_1 = 0.001$ and $\alpha_2 = 0.01$ and $\beta = 40$. Since lifted GUB cover inequalities are already implemented in CPLEX, we assign a random integer value from $[2, 4]$ to K_l for all l. Finally, the density of knapsack constraints is set to 50%, the density of cardinality constraints is set to 3% and the cardinality constraints may overlap. A set of 30 instances of this type is generated for our computational study.

When generating inequalities using sequential lifting, we apply the cardinality-wise lifting algorithm described in [1]. Observe that it is necessary to obtain a set of disjoint cardinality constraints before lifting. To identify a set of disjoint cardinality constraints that are tight, we first sort the cardinality constraints in a non-decreasing order of their ratios $\frac{K_l}{N_l}$. We then select the cardinality constraints with smallest ratios until all the variables are covered. These cardinality constraints are next relaxed so that they do not overlap. As a result, for each instance, we obtain a set of disjoint cardinality constraints on which we apply cardinality-wise lifting. When generating inequalities using sequence-independent lifting, it follows from (39) in Theorem 14 that lifting coefficients will be larger when a_{i^*} is smaller. Therefore, if a variable appears in multiple cardinality constraints, we dynamically assign it to a cardinality set so that a_{i^*} will be smallest. Next, we give a basic description of our cutting plane procedure.

We first solve the LP relaxation of an instance and then apply the separation method described in [16] to find the violated cover inequality. Once such an inequality is found, it is lifted using the sequential lifting algorithm [1] and using the sequence-independent lifting method proposed in this paper. Then, the lifted inequalities are added to the formulation. We repeat the procedure until we cannot generate violated cuts. At this point, we pass the problem to CPLEX. Since lifted cover inequalities obtained using these algorithms sometimes reduce to traditional knapsack cover inequalities, we allow CPLEX to generate cover and GUB cover cuts in all computational experiments. However, we disable other CPLEX cuts that are not using the basic structure we use. Therefore, our results evaluate the improvement obtained by considering cardinality constraints instead of discarding them in the generation of cover cuts. Our experiments are performed with CPLEX 10.1 on a 64-bit Sun workstation with 16G memory.

In Table 1, we present our computational results for each of our testing instances. We also give the average of these measures over all testing instances for comparison. The column labeled CPX denotes the results obtained with CPLEX alone; the column marked SEQ denotes the case where cuts obtained from sequential lifting are added and the column labeled SI denotes the case where cuts obtained from sequence-independent lifting are added. The gap is computed as $100 \times (z^B - z^*)/(z^{LP} - z^*)$ for CPX, SEQ and SI where z^{LP} is the value of LP relaxation without cuts, z^B is the optimal LP value before branching and z^* is the optimal MIP value. To evaluate the efficiency of cut generation, we use *cut time* to record the time spent on generating cuts by sequential and by sequence-independent lifting respectively.

In Table 1, we observe that cuts from sequence-independent lifting might be effective at reducing the number of nodes in branch-and-cut trees as well as the computation time (a reduction of 30% is observed on our randomly generated instances). The cuts from sequential lifting, however, do not seem to yield such computational improvements. A possible explanation is that sequence-independent lifting is performed by dynamically constructing disjoint cardinality constraints while sequential lifting is performed based on a set of pre-defined disjoint cardinality constraints. Therefore, we believe that the difference in performance can be explained by the fact that the dynamic construction of cardinality constraints is more effective at identifying constraints that yield stronger cuts. A natural question for future research is therefore to investigate how to dynamically and efficiently select a set of disjoint cardinality constraints to perform cardinality-wise lifting. Another interesting observation is that the generation of cuts using both cardinality-wise sequential lifting and sequence-independent lifting can be completed quickly. It is of particular interest to note that the computational time spent on sequence-independent lifting is negligible (less than 0.005% of the total computation time) and the overall solution time is reduced by 30%.

7. Conclusion

In this paper, we study the set of 0–1 solutions to a knapsack problem and a set of disjoint cardinality constraints. This model is a generalization of the traditional 0–1 knapsack set and the 0–1 knapsack set with generalized upper bounds. We derive a family of strong valid inequalities for this set by lifting the generalized cover inequalities introduced in [1]. The

Table 1
Computational results.

ID	MIP	LP	CPX			CPX+SEQ				CPX+SI			
			Gap	Nodes	Time	Gap	Nodes	Cut time	Time	Gap	Nodes	Cut time	Time
1	−500	−501.012	97.167	14 865	23.042	91.84	10 588	1.054	16.212	96.294	24 524	0.087	35.37
2	−423	−424.755	94.219	440 226	547.823	94.792	365 218	1.213	470.063	97.375	463 373	0.05	556.596
3	−415	−416.709	95.411	31 733	51.254	97.346	37 275	1.162	66.633	94.154	13 999	0.065	22.317
4	−440	−441.581	96.635	25 567	37.39	96.706	15 605	1.128	24.1	96.348	10 116	0.05	13.678
5	−523	−524.79	97.59	929 853	1 293.468	97.412	856 706	2.268	1 122.553	97.783	670 163	0.059	864.98
6	−467	−468.694	95.272	78 813	105.064	95.798	82 161	0.374	126.927	95.617	35 953	0.061	48.293
7	−399	−400.836	90.716	208 433	303.371	90.398	117 438	2.211	204.95	91.035	109 032	0.142	204.032
8	−512	−512.885	81.644	64 821	92.548	81.67	31 644	1.29	55.06	81.671	72 105	0.047	112.483
9	−426	−427.741	94.598	174 175	194.632	93.015	145 371	2.038	180.472	93.015	192 988	0.098	256.286
10	−422	−423.781	94.645	132 589	173.686	93.798	83 612	0.863	104.169	94.412	79 909	0.055	99.437
11	−430	−430.749	84.722	64 044	83.856	84.222	303 725	1.855	443	84.222	120 571	0.095	162.661
12	−452	−453.286	91.052	85 903	120.129	91.175	300 859	0.818	480.757	92.839	96 459	0.056	161.94
13	−380	−381.287	95.232	46 633	74.95	95.279	164 279	1.444	281.777	95.279	66 813	0.065	108.199
14	−442	−443.983	96.861	549 250	772.913	96.87	437 047	0.724	608.819	96.87	476 263	0.06	683.283
15	−435	−435.778	87.971	12 796	16.219	89.123	7 290	1.133	10.045	88.617	3 413	0.047	4.607
16	−468	−469.732	95.44	248 430	303.706	95.458	324 379	0.724	411.226	95.458	320 584	0.047	397.646
17	−539	−541.163	98.507	3 903 013	5 893.694	98.653	5 059 293	0.613	7 640.575	98.618	6 623 528	0.063	10 113.21
18	−479	−481.078	97.017	39 221 619	58 320.7	96.935	40 623 867	0.205	61 927.83	96.964	20 717 943	0.071	30 171.82
19	−516	−517.818	94.173	114 520	155.563	93.623	143 376	2.218	208.171	93.637	159 472	0.073	234.986
20	−472	−472.763	90.448	648 680	918.981	90.53	284 086	0.697	428.433	90.53	2 891 049	0.047	3 854.512
21	−483	−484.598	97.555	172 192	276.028	97.467	590 828	0.808	1 097.155	97.622	119 575	0.071	199.803
22	−509	−510.427	99.055	23 033	34.081	99.21	26 740	0.648	41.702	98.946	23 094	0.062	33.434
23	−509	−510.558	96.849	69 836	113.606	96.445	70 836	0.809	121.762	96.445	131 842	0.058	235.087
24	−455	−455.944	92.617	12 027	15.707	93.551	26 922	1.773	41.189	93.184	49 874	0.05	84.238
25	−434	−435.483	93.595	120 562	194.419	92.483	46 803	1.612	76.83	92.27	11 938	0.12	22.85
26	−398	−399.601	95.332	1 592 896	2 553.091	95.187	6 175 867	0.239	9 903.117	97.081	330 251	0.075	553.152
27	−464	−464.886	94.298	106 025	144.32	94.064	29 811	1.059	42.411	94.09	85 320	0.054	127.205
28	−390	−392.109	95.633	2 463 832	2 878.286	95.571	2 172 979	1.564	3 124.387	95.551	1 985 663	0.132	2 931.058
29	−479	−479.773	98.247	1 611	1.714	97.877	9 697	0.797	10.883	97.877	2 912	0.044	3.02
30	−421	−422.998	94.676	334 786	449.768	94.64	236 516	0.873	376.458	94.64	255 306	0.055	383.858
Avg.	NA	NA	94.24	1 729 759	2 538.13	94.04	1 959 361	1.14	2 987.78	94.28	1 204 801	0.07	1 756.04

lifting is performed using a set of multi-dimensional superadditive lifting functions that are proven to be non-dominated and maximal. The lifted generalized cover inequalities that we obtain dominate those derived using the knapsack constraint only. Therefore, our results strictly generalize the classical results about lifted covers. This is, to the best of our knowledge, the first work in which multi-dimensional superadditive approximations of lifting functions are proven to be non-dominated and maximal. Our computational results also suggest that the inequalities generated by sequence independent lifting from multiple constraints could be useful in branch-and-cut algorithms. Therefore, we believe that multi-constraint lifting is a computational tool to generate stronger cuts that could yield improvements in general purpose MIP solvers and that should be investigated further.

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